Extremal principles and limiting possibilities of open thermodynamic and economic systems

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Abstract. In this paper Prigogine's minimum entropy principle is generalized to thermodynamic and microeconomic systems that include active subsystem (heat engine or economic intermediary). New bounds on the limiting possibilities of an open system with active subsystem are derived, including the bound on the productivity of the heat-driven separation. The economic analogies of Onsager's reciprocity conditions are derived.

1. Introduction

Thermodynamic and microeconomic systems are both *macro-systems.* They include a large number of micro subsystems, which are not controllable and not observable. Control and observation in such systems is only feasible on the macro level. The state of macro-system is described by macro variables that depend on the averaged behavior of its components only. Macro-variables are divided into extensive and intensive. The former include internal energy, entropy, mass in thermodynamics and stocks of resources and capital in economics. When a system is subdivided its extensive variables change proportionally to the volume. The intensive variables (pressure, temperature, chemical potential in thermodynamics, resource's and capital's estimates in microeconomics) do not change if the system is subdivided. In equilibrium macro system's variables are linked via the equation of state.

Changes of extensive variables are linked to flows of mass, energy, resources, capital, etc. These flows can be caused by external factors (convective flows) or by interaction of macro-systems with each other (in thermodynamics such flows are called diffusive). Flow rates depend on the differences between the intensive variables of the interacting systems. This exchange leads to changes of extensive variables. The rate of their change is proportional to the exchange flows. It is useful to single out three classes of macro-systems:
1. *Systems with infinite capacity (reservoirs).* The values of their intensive variables are fixed and do not depend on the exchange flows.
2. *Finite capacity systems.* We assume that they always are in internal

equilibrium. Their intensive variable change as a result of exchange flows. For example, the temperature of the system with constant volume changes if its internal energy changes, the price of a resource changes when its stock changes, etc. If interacting subsystems of finite capacity are insulated from the environment then the differences of their intensive variables tend to zero and the system as a whole tends to its equilibrium state.
3. System with intensive variables (all or some) that can be controlled (within given range). We shall call such systems *active*. Working body of the heat engine, whose parameters are controlled to achieve maximal performance, is an active system. An economic intermediary, who buys and resells resource by offering one price for buying and another for reselling, is an active system. Active systems play important role in Finite-Time Thermodynamics (FTT), which investigates limiting possibilities of non-equilibrium thermodynamic systems (Berry, 1999), (Salamon, 2001),(Andresen, 1983), (Andresen, 1984), (Bejan, 1996). Most of FTT problems are reduced to optimal control problems where intensive variables of active systems are controls.

A macro-system is *open*, if it exchanges with its environment. If environment includes reservoirs and some of the flows are convective then its steady state can be stationary (system's intensive variables are time-constant), periodic, quasi-periodic or quasi-stochastic.

Near equilibrium the flows in a system depend linearly on the driving forces. Prigogine's extremal principle states that *steady state of near equilibrium system is stationary and the values of its intensive variables are distributed within its volume or between its finite capacity subsystems is such a way that the entropy production in the system is minimal* (Prigogine, 1971).

We will consider open thermodynamic and microeconomic macrosystems that include internally equilibrium subsystems of these three types. We will obtain the conditions that determine the limiting possibilities of active systems here and the extremal principles that determine stable states of such systems.

2. Thermodynamic system that includes an active subsystem

2.1. Problem formulation

We consider thermodynamic system that includes n finite-capacity subsystems (we shall call them subsystems) ($i = 1, ..., n$) in internally-equilibrium and an active subsystem that transforms heat or chemical energy into work. We shall call this active subsystem a transformer. System exchanges convective mass flows g_k $(k = 1, ..., r)$ with the

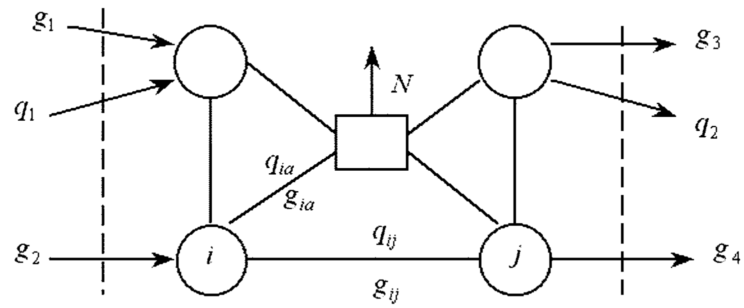

Figure 1. Open thermodynamic system with transformer.
Figure 0.

environment. The compositions and rates of some of these flows are given. The flows of heat q_k are also given. Mass and heat flows between subsystems inside the system are denoted as (q_{ij}, g_{ij}) and between subsystems and transformer as q_{ia}, g_{ia}. We assume that these flows are linear on driving forces Δ_{ij}

$$\left. \begin{array}{l} G_{ij} = \{q_{ij}, g_{ij}\} = L_{ij}\Delta_{ij}, \\ G_{ia} = \{q_{ia}, g_{ia}\} = L_{ia}\Delta_{ia}. \end{array} \right\} \quad (1)$$

L_{ij} and L_{ia} are the matrices of kinetic coefficients. We consider the flow that enters a subsystem (convective and diffusion) as positive and the power N, produced by the transformer, as positive also. If $N > 0$, then we shall call the transformer direct, and if $N < 0$ then it is inverse. Driving forces Δ_{ia} depend on the state of the transformer during contact with i-th subsystem. It can be controlled. The state of the i-th subsystem is described by the vector of its intensive variables y_i. Here $T_i = \frac{1}{y_{i0}}$,$\Delta_{ij} = (y_i - y_j)$, $T_{ia} = \frac{1}{y_{ia0}}$, $\Delta_{ai} = (y_a - y_i)$. We assume that some of subsystems have given state $(x_i = x_i^0, i = 1, ..., m \leq n)$, and the other are free.

The following problems can be formulated in relation to this system:
1. What are the values of free variables x_i $(i = m+1, ..., n)$ if the power of the transformer N and the compositions and rates of convective flows are fixed?
2. What is the maximal power that can be extracted the and what is the minimal power that has to be spent N (the limiting power problem)?
3. What is the maximal-feasible rate of one of the flows g_r (objec-

tive flow) if the power N and the compositions of all or some of the convective flows are fixed?

2.2. Thermodynamic balances

Let us write down thermodynamic balances for a steady state of the system as a whole. Subscript k denotes convective flow k. $e_k, V_k, h_k = e_k + p_k V_k, p_k$ denote k-th convective flow's molar energy, enthalpy, pressure, and its internal energy as e_k.
(a) Energy balance is

$$\sum_{k=0}^{r}(g_k h_k + q_k) - N = 0. \tag{2}$$

(b) Mass balance is

$$\sum_{k=0}^{r} g_k x_{\nu k} + \sum_{l} \alpha_{\nu l} W_l = 0, \quad \nu = 1, 2, \ldots \tag{3}$$

$$\sum_{\nu} x_{\nu k} = 1, \quad k = 1, \ldots, r. \tag{4}$$

Here $x_{\nu k}$ is the molar fraction of the ν-th component in the k-th flow, $\alpha_{\nu l}$ is stoichiometric coefficient with which ν-th specie enters into equation for the l-th reaction, W_l is the rate of this reaction. To simplify equation we assume that the flow of energy from the reaction

$$q_l = \sum_{\nu} \mu_{\nu l} \alpha_{\nu l} W_l, \tag{5}$$

is included into convective heat flows q_k.
Similarly we introduce flows of mass

$$g_l = W_l \sum_{\nu} \alpha_{\nu l}, \tag{6}$$

with the composition determined by the equality

$$x_{\nu l} = \frac{\alpha_{\nu l}}{\sum_{\nu} \alpha_{\nu l}}. \tag{7}$$

We include these flows into convective flows g_k.
(c) Entropy balance is

$$\sum_{k=0}^{r}\left(g_k s_k + \frac{q_k}{T_k}\right) + \sigma = 0. \tag{8}$$

Here s_k is the molar entropy of the k-th flow. $g_k > 0$, if it enters and $g_k < 0$, if it leaves the system.
From (3), (4) it follows that $\sum\limits_{k=0}^{r} g_k = 0$, so we eliminate one of the flows (objective flow) g_0. We obtain

$$\sum_{k=0}^{r} q_k + \sum_{k=1}^{r} g_k \Delta h_{0k} - N = 0, \tag{9}$$

$$\sum_{k=0}^{r} \frac{q_k}{T_k} + \sum_{k=1}^{r} g_k \Delta s_{0k} + \sigma = 0. \tag{10}$$

Here

$$\Delta h_{0k} = h_k - h_0, \quad \Delta s_{0k} = s_k - s_0. \tag{11}$$

Elimination of q_0 with the temperature T_0 from the equation (9) and its substitution into (10), yields

$$\sum_{k=1}^{r} \left[g_k \left(\Delta s_{0k} - \frac{\Delta h_{0k}}{T_0} \right) + q_k \left(\frac{1}{T_k} - \frac{1}{T_0} \right) \right] + \sigma + \frac{N}{T_0} = 0. \tag{12}$$

We denote the efficiency of the reversible heat engine as

$$\eta_C^0 = \frac{T_k - T_0}{T_k}$$

The power of the transformer N can be then expressed from (12) as

$$N = \sum_{k=1}^{r} [q_k \eta_C^0 + g_k(\Delta h_{0k} - \Delta s_{0k} T_0)] - T_0 \sigma = N^0 - T_0 \sigma. \tag{13}$$

The first term in the right-hand side of the equality is the reversible power N^0 in a system with infinitely large mass and heat transfer coefficients (arbitrary large size of apparatus). It is completely determined by the parameters of the system's input and output convective flows. The second term describes dissipative losses.
For mixtures, which are close to ideal gases or ideal solutions, the molar enthalpies and entropies can be expressed in terms of their compositions

$$h_k(T_k, p_k, x_k) = \sum_{\nu} x_{\nu k} h_k(T_k, p_k), \tag{14}$$

$$s_k(T_k, p_k, x_k) = \sum_{\nu} x_{\nu k} [\overline{s}_k(T_k, p_k) - R \ln x_{\nu k}], \quad k = 0, ..., r. \tag{15}$$

From the equality (13) it follows that if transformer's intensive variables are chosen in such a way that N^0 is not changed then the maximum of N is achieved when minimum of σ is achieved.

2.3. Entropy production and the state of subsystems

For Onsanger's linear kinetic (1) the entropy production in the system takes the following form

$$\sigma = \sum_{i=1}^{n} \left(\Delta_{ia}^T L_{ia} \Delta_{ia} + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq i}}^{n} \Delta_{ij}^T L_{ij} \Delta_{ij} \right). \tag{16}$$

The first term is the entropy production (scalar product of fluxes on driving forces) due to heat and mass exchange between the i-th subsystem and transformer. The second term is the entropy production due to heat exchange between i-th and j-th subsystems. Multiplier $\frac{1}{2}$ appears because the term for each flow enters into equality (16) twice. Matrices L_{ij} and L_{ia} are positive definite and symmetric.
First we assume that convective flows do not enter into subsystems with non-fixed state $y_i (i = m+1, ..., n)$. The following analog of Prigogine's principle then holds for an open system that includes a transformer (active subsystem)
Statement 1: *If intensive variables of the transformer $y_{ia} (i = 1, ..., n)$ are fixed then the free intensive variables of subsystems y_i of an open system take such values that entropy production in the system is minimal and the power N obeys the equation*

$$N = N^0 - \sigma_{\min}(N) T_0. \tag{17}$$

The proof of this statement follows from the fact that stationarity conditions of σ with respect to the components of the state vector y_i of the i-th subsystem coincide with the condition of its minimum, because σ is a convex function.
Indeed, change of y_i effects all flows that enter/ leave the i-th subsystem. Note that the derivatives $\frac{\partial \Delta_{ij}}{\partial y_i}$ and $\frac{\partial \Delta_{ia}}{\partial y_i}$ have opposite signs and their absolute values are equal 1. Since matrices L_{ij} and L_{ia} are symmetric the stationarity conditions lead to the equations

$$\sum_{j=1, j \neq i}^{n} g_{ij\nu} = g_{ia\nu}, \quad i = m+1, ..., n, \quad \nu = 1, 2, ... \tag{18}$$

where $g_{ij\nu} = g_{ij}x_{j\nu}$, $g_{ia\nu} = g_{ia}x_{i\nu}$ are the diffusion flows of the ν-th component

$$\left(\sum_{\nu} g_{ij\nu} = g_{ij}, \sum_{\nu} g_{ia\nu} = g_{ia} \right),$$

$$\sum_{j=1, j\neq i}^{n} (q_{ij} + g_{ij}h_{ij}) = \begin{array}{c} q_{ia} + g_{ia}h_i \\ i = 1, ..., n. \end{array} \tag{19}$$

These equations coincide with equations of mass and energy balances for the i-th subsystem.
Thus the intensive variables of the subsystems of an open system with a transformer take such values that dissipation in it is minimal. If some of the intensive variables y_i are fixed then the free variables minimize σ subject to these constraints.

2.4. Limiting power problem

If the state of some of the subsystems and the parameters of the convective flows are fixed then the power N, obtain from the system or used to maintain its state is bounded. For $N > 0$ this bound is the maximal power of heat engine (Novikov, 1957), and for $N < 0$ it is the minimal losses in the transformer (Tsirlin , 2003)).
Assume that convective flows q_{ka} and g_{ka} are equal zero. Then the thermodynamic balances take the following form:
(a) Energy balance is

$$N = \sum_{i=1}^{n} \left(q_{ia} + \sum_{\nu} g_{ia\nu}h_{i\nu} \right). \tag{20}$$

(b) Mass balance is

$$\sum_{i=1}^{n} g_{ia\nu} = 0, \quad \nu = 1, 2, ... \tag{21}$$

(c) Entropy balance is

$$\sum_{i=1}^{n} \left[g_{ia}s_i + \frac{1}{T_{ia}} \left(q_{ia} + \sum_{\nu} g_{ia\nu}h_{i\nu} \right) \right] = 0. \tag{22}$$

The problem of limiting power is now reduced to such choice of transformer's variables y_{ia} during its contact with the i-th subsystem for which N is maximal subject to conditions (21), (22), and to the mass, energy and entropy balances for each of the i-th subsystems. This formulation of the problem is very general. We will consider two particular cases of this problem.

2.4.1. *Heat-mechanical system*

Direct transformer. Here there are no mass flows and the problem can be rewritten as follows ($q_{ia} \sim q_i$)

$$N = \sum_{i=1}^{n} q_i(T_i, T_{ia}) \to \max_{T_{ia}, T_i} \tag{23}$$

subject to entropy and energy balances

$$\sum_{i=1}^{n} \frac{q_i(T_i, T_{ia})}{T_{ia}} = 0, \tag{24}$$

$$\sum_{j=1}^{n} q_{ij}(T_i, T_j) + \sum_{k=0}^{r} q_{ik} = q_{ia}, \quad i = 1, ..., n. \tag{25}$$

The conditions of optimality for the problem (23)–(25), follows from the conditions of stationarity of its Lagrange function

$$L = \sum_{i=1}^{n} \left\{ q_i \left(1 + \frac{\Lambda}{T_{ia}} - \lambda_i \right) + \lambda_i \left(\sum_{j=1}^{n} q_{ij} + \sum_{k=1}^{r} q_{ik} \right) \right\} \tag{26}$$

on T_{ia} and T_i

$$\frac{\partial L}{\partial T_{ia}} = 0 \Rightarrow \frac{\partial q_i}{\partial T_{ia}} \left(1 + \frac{\Lambda}{T_{ia}} - \lambda_i \right) = \Lambda \frac{q_i(T_i, T_{ia})}{T_{ia}^2}, \quad i = 1, ..., n, \tag{27}$$

$$\frac{\partial L}{\partial T_i} = 0 \Rightarrow \frac{\partial q_i}{\partial T_i} \left(1 + \frac{\Lambda}{T_{ia}} - \lambda_i \right) + \lambda_i \sum_{j=1}^{m} \frac{\partial q_{ji}}{\partial T_i} = 0, \quad i = m+1, ..., n. \tag{28}$$

Equations (27), (28) allows us to find n temperatures T_{ia} of the working body, $(n - m)$ temperatures T_i of subsystems and $(n + 1)$ Lagrange multiplier.

The most common form of heat exchange law is Newton's linear form

$$q_{ji} = \alpha_{ji}(T_j - T_i).$$

Here $q_i = \alpha_i(T_i - T_{ia})$, $q_{ji} = \alpha_{ji}(T_j - T_i)$, and equations (27), (28) take the form

$$\sum_{i=1}^{n} \overline{\alpha_i} \frac{T_i}{T_{ia}} = 1, \qquad \overline{\alpha_i} = \frac{\alpha_i}{\sum_{\nu=1}^{n} \alpha_\nu}, \tag{29}$$

$$T_{ia} = T_i - \frac{1}{\alpha_i} \left[\sum_{j=0}^{n+1} \alpha_{ji}(T_j - T_i) + \sum_k q_{ik} \right], \quad i = 0, ..., n+1, \tag{30}$$

$$T_{ia}^2(1-\lambda_i) = \Lambda T_i, \quad i = 1, ..., n, \tag{31}$$

$$\alpha_i \left(1 + \frac{\Lambda}{T_{ia}} - \lambda_i\right) = \lambda_i \sum_{j=1}^{n} \alpha_{ji}, \quad i = 1, ..., n. \tag{32}$$

We denote

$$\overline{\alpha_i} = \frac{\alpha_i}{\sum\limits_{j=0}^{m+1} \alpha_{ji}}.$$

After eliminating λ_i from (31) and (32) we get

$$(1 + \overline{\alpha_i})\frac{T_i}{T_{ia}^2} + \frac{\overline{\alpha_i}}{T_{ia}} = \frac{1}{\Lambda} = \text{const}, \quad i = 1, ..., n. \tag{33}$$

These equations jointly with balances (29), (30) determine the solution.

Let us show that if $n = 2$ and subsystems' temperatures are fixed $T_1 = T_+, T_2 = T_-$ then from these conditions follow the known results (Novikov, 1957), (Curzon , 1975) about the limiting power of the heat engine. Indeed, here the conditions (32) are missing, $\lambda_i = 0$ and from (31) it follows that

$$T_{1a}^* = \sqrt{\Lambda T_+}, \quad T_{2a}^* = \sqrt{\Lambda T_-},$$

and from equations (29)–(32) obtain the efficiency of the heat engine with maximal power as

$$\eta = 1 - \frac{T_{0a}}{T_{1a}} = 1 - \sqrt{\frac{T_-}{T_+}},$$

and the maximal power as

$$N_{\max} = \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}(\sqrt{T_+} - \sqrt{T_-})^2. \tag{34}$$

2.4.2. *Inverse transformer. Optimal thermostating.*
The maximal feasible power can be positive or negative. The sign depends on the given temperatures of the external flows q_k and given fixed temperatures of the subsystems $T_i (i = 1, ..., m)$. If this power is negative then we obtain the problem of limiting possibilities of heat pump in the thermo-stating system. That is, the problem of maintaining given temperatures in some of the subsystems and of optimal choice of temperatures in the rest of passive subsystems to minimize power used. For Newton laws of heat transfer we obtain the conditions (29)–(31). The conditions (32) and (33) hold for subsystems with free

temperatures T_i $(i = m+1, ..., n)$. In the optimal thermo-stating problem (Tsirlin , 2003) these temperatures are the temperatures of the passive subsystems.

2.5. Separation system

Separation systems commonly used mechanical (membrane systems, Centrifuging, etc) or heat energy (distillation, drying, etc). We consider them separately.

2.5.1. *Binary separation using mechanical energy*

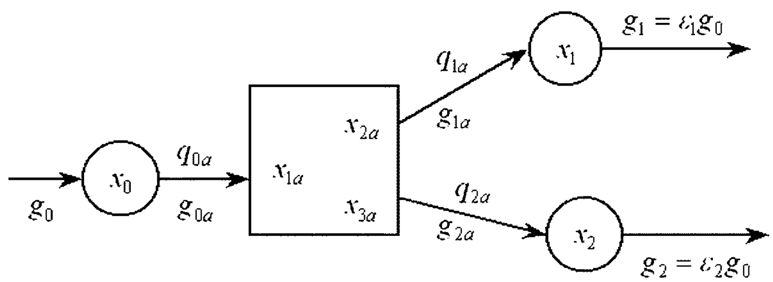

Figure 2. Mechanical separation of binary mixture.
Figure 0.

Consider system for binary separation shown in Figure 2. The input points for convective flows are fixed and the compositions and rates of these flows are also fixed. Here the mass balance holds

$$g_1 = g_2 + g_3, \tag{35}$$

$$g_1 x_{1\nu} = g_2 x_{2\nu} + g_3 x_{3\nu}, \quad \nu = 1, 2, ..., \tag{36}$$

The composition of each of three subsystems $x_i (i = 1, 2, 3)$ are also fixed and coincide with the compositions of the convective flows. The temperatures and pressures of these flows are the same and enthalpy increments of the convective flows are equal zero. The condition (13) can be rewritten as

$$N = -g_0 T_0 \sum_{k=1}^{1} \varepsilon_k \Delta s_{0k} - T_0 \sigma \to \max . \tag{37}$$

Here $\varepsilon_k = \left|\frac{g_k}{g_0}\right|$ is the fraction removed into the first and the second flows ($\varepsilon_1 + \varepsilon_2 = 1$). We denote the power used as $\overline{N} = -N$ and obtain

$$\overline{N} = g_0 R T_0 \left[\sum_{k=1}^{2} \varepsilon_k \sum_{\nu} x_{k\nu} \ln x_{k\nu} - \sum_{\nu} x_{0\nu} \ln x_{0\nu}\right] + T_0\sigma = \overline{N^0} + T_0\sigma \to \min . \tag{38}$$

The first term in the right hand side is the reversible power for separation.

Each of the convective flows $\overline{g}_i$ contains three components: $q_i, g_{i1} = g_1 x_{i1}, g_{i2} = g_1 x_{i2}$ ($i = 0, 1, 2$). The vector of driving force of the i-th flow Δ_i has components

$$\Delta_{iq} = \left(\frac{1}{T_{ia}} - \frac{1}{T_0}\right), \quad \Delta_{i1} = \left(\frac{\mu_{i1a}}{T_{ia}} - \frac{\mu_{i1}}{T_0}\right), \quad \Delta_{i2} = \left(\frac{\mu_{i2a}}{T_{ia}} - \frac{\mu_{i2}}{T_0}\right).$$

Flows depend linearly on driving forces

$$\overline{g}_i = A_i \Delta_i, \quad i = 0, 1, 2, \tag{39}$$

here the matrix A_i is positive definite and symmetric. Its inverse $B_i = A_i^{-1}$ is also positive definite and symmetric.
Entropy production can be rewritten using (39) in the following form

$$\sigma = \sum_{i=0}^{2} \sigma_i = \sum_{i=0}^{2} \overline{g}_i^T A_i^{-1} \overline{g}_i. \tag{40}$$

If it is feasible to control driving forces in the system in such a way that the rates and compositions of flows have required values then the power used by the irreversible separation system is

$$\overline{N}^* = \overline{N}^0 + T_0 \sum_{i=0}^{2} \overline{g}_i^T A_i^{-1} \overline{g}_i. \tag{41}$$

In particularly for diagonal matrices A_i

$$\overline{N} = \overline{N}^0 + T_0 \sum_{i=0}^{2} \left(\frac{q_i^2}{\alpha_{iq}} + \frac{g_{i1}^2}{\alpha_{i1}} + \frac{g_{i2}^2}{\alpha_{i2}}\right). \tag{42}$$

If the constrains imposed on the system are softer, for example, if only the flow g_1 and its composition $x_{11} (x_{12} = 1 - x_{11})$ and the composition of the input flow $x_{01} (x_{02} = 1 - x_{01})$ are given then the problem is reduced to the search of minimum of σ on $q_0, q_1, q_2, g_0, g_2, g_{21}$ subject to constraints

$$\sum_{i=0}^{2} q_i = \sum_{i=0}^{2} g_i = \sum_{i=0}^{2} g_{i1} = 0, \quad 0 \le g_{21} \le g_2. \tag{43}$$

Quadratic form (40) is convex and the set of this problems feasible solutions is also convex. Therefore the problem (40), (43) has a unique solution which corresponds to the power that is always lower than $\overline{N}^*$, found from (41).

Note that productivity of the system (rate of objective flow) can be made arbitrary high if the power used $\overline{N}$ is made sufficiently high.

2.6. Thermal separation

Here $N = 0$ and the transformer uses heat flows to obtain the work of separation. Consider the system shown in Figure 3

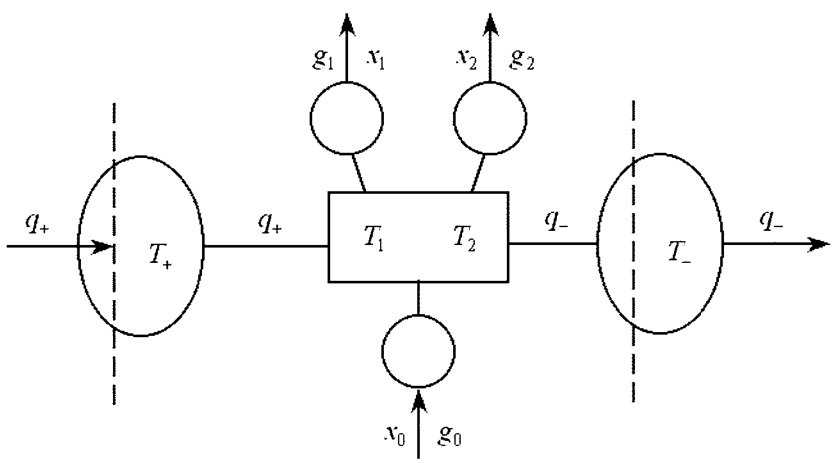

Figure 3. Thermal separation system.
Figure 0.

Assume that the input mixture is binary and the temperatures of the flows $g_i (i = 0, 1, 2)$ and their pressures are the same. We denote the temperatures of the subsystems as T_+ and $T_- < T_+$. We assume that enthalpy increments of the flows g_1 and g_2 are zero, $N = 0$, and the formula (13) takes the form

$$q_+\eta = T_- \left(\sum_{k=1}^{2} g_k \Delta s_{0k} + \sigma \right) = 0, \quad \eta = 1 - \frac{T_-}{T_+}. \tag{44}$$

For Newton law of heat transfer

$$q_+ = \alpha_+(T_+ - T_1), \quad q_- = \alpha_-(T_- - T_2), \tag{45}$$

and for the substances that are close to ideal solutions the reversible power of separation is

$$N_p(g_0) = T_0 \sum_{k=1}^{2} g_k \Delta s_{0k} = RT_0 g_0 \left(\sum_{k=1}^{2} |\varepsilon_k| \sum_{\nu=1}^{2} x_{k\nu} \ln x_{k\nu} - \right. \\ \left. - \sum_{\nu=1}^{2} x_{0\nu} \ln x_{0\nu} \right), \quad \varepsilon_k = \frac{|g_k|}{g_0}. \tag{46}$$

The entropy production due to heat flows is

$$\sigma_q = \sigma_{q_+} + \sigma_{q_-} = \frac{\alpha_+(T_+ - T_1)^2}{T_+T_1} + \frac{\alpha_-(T_- - T_2)^2}{T_-T_2}. \tag{47}$$

The entropy production due to mass flows is (40)

$$\sigma_g = \sum_{k=0}^{2} \overline{g_k}^T A_k^{-1} \overline{g}_k. \tag{48}$$

After taking into account (45)–(48) the equation (44) takes the form

$$\alpha_+(T_+ - T_1)\eta = (N_p + T_-\sigma_g) + T_-\sigma_q. \tag{49}$$

Note that for the fixed ε_k and fixed compositions of flows the reversible power of separation N_p increases linearly when productivity g_0 increases, and that σ_g increases as g_0^2. Therefore the maximal productivity of thermal separation system $g^0_{\max}$ is finite and is given by the solution of the equation

$$N_p(g_0) + T_-\sigma_g(g_0) = \max_{T_1,T_2} \left\{ q_+(T_+, T_1)\eta - \sigma_q(T_+, T_-, T_1, T_2) \right\}. \tag{50}$$

The expression in the right hand side of the equality (16) is the maximal power of irreversible heat engine. Its maximum is sought subject to transformer's balance on the entropies of heat flows

$$\frac{\alpha_+(T_+ - T_1)}{T_1} + \alpha_- \frac{(T_- - T_2)}{T_2} = 0. \tag{51}$$

The solution of this problem for Newton laws of heat transfer was obtained in (Novikov, 1957), (Curzon , 1975) and shown above (34). Substitution of $N_{\max}$ into the right hand side of equality (50) allows us to obtain the limiting productivity of thermodynamic separation system that has the structure shown in Figure 3.

Let us emphasis that unlike mechanical systems the productivity of the thermal systems is bounded. The increase of the rates of heat flows above some threshold reduces system's.

3. Open microeconomic system

The analogy between thermodynamic and microeconomic systems has been studied extensively. The works Samuelson (Samuelson , 1972), Lichnerowicz (Lichnerowicz, 2003), Rozonoer (Rozonoer, 1971), (Rozonoer, 1973) and Martinash (Martinas, 2003) should be especially mentioned. Most of that research considered analogy between equilibrium systems. In this paper we consider this analogy for non-equilibrium systems.

3.1. Stationary state, reciprocity conditions and minimal dissipation principle

Each subsystem on an open economic system - an economic agent - is described by its extensive variables - the stock of resources N and capital N_0; by its wealth function $S(N)$, and by its intensive variables - resources' and capital estimates p_i and p_0 that obey the following equations

$$p_0 = \frac{\partial S}{\partial N_0}, \quad p_i = \frac{\partial S}{\partial N_i} \Big/ \frac{\partial S}{\partial N_0} = \frac{1}{p_0} \frac{\partial S}{\partial N_i}, \quad i = 1, 2, ... \tag{52}$$

Resource estimate p_i is the equilibrium price for buying and selling. If the price c_i is higher than the equilibrium price then economic agent sells and if it is lower then it buys resource. Because agent's wealth function is uniform function of first degree and strictly convex, the resource's estimate decreases when its stock increases.

If the system is near equilibrium then the flow depend on the driving forces linearly. The driving force for the i–th resource is the difference between its price and its estimate $\Delta_i = p_i - c_i$. We assume that it is positive if the flow is directed to the economic agent, then

$$g_i = \sum_{\nu=1}^{n} a_{\nu i} \Delta_\nu = \sum_{\nu=1}^{n} a_{\nu i}(p_\nu - c_\nu), \quad i = 1, ..., n \tag{53}$$

We shall call the matrix A with elements $a_{i\nu}$ the matrix of economic agent's kinetic coefficient. This matrix determines exchange kinetic between economic agent and its environment.

The flow of resource exchange causes the reciprocal flow of capital in the opposite direction such that

$$\frac{dN_0}{dt} = -\sum_{i=1}^{n} c_i g_i. \tag{54}$$

Economic agent's wealth function here changes

$$\frac{dS}{dt} = \frac{\partial S}{\partial N_0}\frac{dN_0}{dt} + \sum_{i=1}^{n} \frac{\partial S}{\partial N_i} g_i = -p_0 \sum_{i=1}^{n} c_i g_i + p_0 \sum_{i=1}^{n} p_i g_i = p_0 \sum_{i=1}^{n} (p_i - c_i) g_i = p_0 \Delta^T A \Delta. \quad (55)$$

Here Δ is the driving force vector.

Because the estimate of the capital $p_0 > 0$, and because the exchange is always done voluntarily, the wealth function does not decrees. Therefore the matrix A is positive definite. If the driving forces are expressed in terms of Δ from (53) then (55) takes the form

$$\frac{dS}{dt} = p_0 g^T B g, \quad (56)$$

here g is column-vector, $B = A^{-1}$, and the elements $b_{i\nu}$ of this matrix are equal to (up to the constant)

$$b_{i\nu} = \frac{\partial^2 S}{\partial N_i \partial N_\nu}, \quad i, \nu = 1, ..., n. \quad (57)$$

Thus, the matrix B is positive definite and symmetric, and its inverse matrix of kinetic coefficients A is also positive definite and symmetric. Thus, the following analog of the reciprocity conditions hold: *the effect of the difference between the price and estimate of the ν-th resources on the flow of i-th resource is the same as the effect of the difference between the price and estimate of the i-th resources on the flow of ν-th resource.*

Consider the system that includes r subsystems with fixed resource estimates (economic reservoirs) and $k - r$ subsystems that exchange resources and capital with each other (Figure 4). Resource estimates for reservoirs p_i $(i = 1, \ldots, r)$ are constant. These estimates N_i and N_{0i} depend on the stocks of resource and capital for subsystems with $i > r$. We assume for simplicity that capital estimates are the same for all subsystems and equal to 1. Assume that the flow of resource n_i to/from economic agent, who has estimate $p_i(N_i, N_{0i})$, is determined by the flow of capital $q_i = -n_i c_i$. Here c_i is the price of resource that is higher than p_i when economic agent sells resource $(n_i < 0)$, and is lower than p_i when it buys it, $n_i > 0$. If economic agent exchanges with j-th reservoir, then $c_i = p_j$, and

$$n_{ij} = n_{ij}(p_j, p_i), \quad q_{ij} = -p_j n_{ij}, \qquad j = 1, \ldots, r; \quad i = r + 1, \ldots, k. \quad (58)$$

Following (Martinas, 2003), we define the price $c_{i\nu}$ such that

$$\tilde{n}_{i\nu}(p_i, c_{i\nu}) = -\tilde{n}_{\nu i}(p_\nu, c_{i\nu}), \quad (59)$$

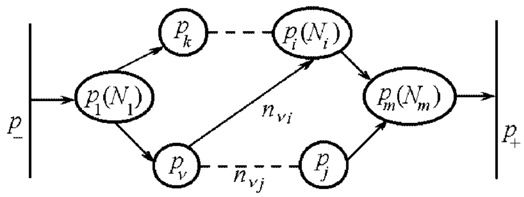

Figure 4. Structure of an open microeconomic system with two economic reservoirs.
Figure 0.

where $(i;\nu) \geq r+1$. Conditions (59) allows us to express $c_{i\nu}$ in terms of p_i, p_ν and obtain

$$n_{i\nu}(p_i, p_\nu) = -n_{\nu i}(p_\nu, p_i). \tag{60}$$

For example, assume that

$$\tilde{n}_{i\nu} = \tilde{a}_{i\nu}(p_i - c_{i\nu}), \quad \tilde{n}_{\nu i} = \tilde{a}_{\nu i}(p_\nu - c_{i\nu}).$$

From the condition (59) we obtain $c_{i\nu}$ and flows in the equation (60) become:

$$c_{i\nu} = \frac{\tilde{a}_{i\nu}p_i + \tilde{a}_{\nu i}p_\nu}{\tilde{a}_{i\nu} + \tilde{a}_{\nu i}} = \gamma_{i\nu}p_i + \gamma_{\nu i}p_\nu, \quad \gamma_{i\nu} + \gamma_{\nu i} = 1, \tag{61}$$

$$\begin{aligned} n_{i\nu}(p_i, p_\nu) &= \frac{\tilde{a}_{\nu i}\tilde{a}_{i\nu}}{\tilde{a}_{i\nu} + \tilde{a}_{\nu i}}(p_i - p_\nu) = a_{i\nu}(p_i - p_\nu), \\ n_{\nu i}(p_\nu, p_i) &= -n_{i\nu}(p_i, p_\nu) = a_{i\nu}(p_\nu - p_i). \end{aligned} \tag{62}$$

The capital fluxes are

$$q_{i\nu}(p_i, p_\nu) = -c_{i\nu}(p_i, p_\nu)n_{i\nu}(p_i, p_\nu) = -q_{\nu i}(p_\nu, p_i). \tag{63}$$

For fluxes (62) we obtain

$$q_{i\nu}(p_i, p_\nu) = -\frac{(\tilde{a_{i\nu}}p_i + \tilde{a_{\nu i}}p_\nu)\tilde{a_{\nu i}}\tilde{a_{i\nu}}}{(\tilde{a_{i\nu}} + \tilde{a_{\nu i}})^2}(p_i - p_\nu).$$

If $\tilde{a_{i\nu}} = \tilde{a_{\nu i}} = \overline{a_{i\nu}}$ then

$$q_{i\nu}(p_i, p_\nu) = -\frac{\overline{a_{i\nu}}(p_i^2 - p_\nu^2)}{4}.$$

Assume that the flux is a vector and the condition (60) holds for l-th component of this flux. We denote:
(a) Vector of differences between estimates of i-th and ν-th economic agents (b)

$$\Delta p_{i\nu} = (\Delta p_{i\nu 1}, ..., \Delta p_{i\nu l}, ...) = p_i - p_\nu. \qquad (64)$$

(b) Matrix $A_{i\nu}$ of the coefficients that link flux-vector between economic agents and their estimates' difference. The elements of this matrix $a_{i\nu\mu l}$, where μ and l are subscripts that denote the type of resource. Matrix $A_{i\nu}$ is positive definite ($A_{i\nu} = 0$) and symmetric.

The flow of l-th resource between i-th and ν-th economic agents is

$$n_{i\nu l} = \sum_\mu a_{i\nu\mu l}(\Delta p_{i\nu\mu}), \qquad (65)$$

and vector-flux is

$$n_{i\nu} = A_{i\nu}\Delta p_{i\nu}. \qquad (66)$$

We denote the price vector during exchange between economic agents as $c_{i\nu} = (c_{i\nu 1}, ..., c_{i\nu l}, ...)$. From the condition similar to (59), it follows that the price is equal to the subsystems' estimates averaged with the weights $\gamma_{i\nu l}$ and $\gamma_{\nu il}$

$$c_{i\nu l}(p_{il}, p_{\nu l}) = \frac{\tilde{a_{i\nu l}} p_{il} + \tilde{a_{\nu il}} p_{l\nu}}{\tilde{a_{i\nu l}} + \tilde{a_{\nu il}}} = \gamma_{i\nu l} p_{il} + \gamma_{\nu il} p_{\nu l}, \qquad (67)$$

The flux of capital is

$$q_{i\nu} = -c_{i\nu} A_{i\nu} \Delta p^T_{i\nu}. \qquad (68)$$

In a steady state the stocks of resources and capital do not change

$$\sum_{\nu=1}^{k} n_{i\nu} = \sum_{\nu=1}^{k} A_{i\nu}\Delta p^T_{i\nu} = 0, \quad i = r+1, ..., k, \qquad (69)$$

$$\sum_{\nu=1}^{k} c_{i\nu}(p_i, p_\nu) A_{i\nu}\Delta p^T_{i\nu} = 0, \quad i = r+1, ..., k. \qquad (70)$$

Capital dissipation is

$$\sigma = \frac{1}{2}\sum_{i,\nu} \Delta p_{i\nu} A_{i\nu} \Delta p^T_{i\nu}, \qquad (71)$$

the multiplier $\frac{1}{2}$ is due to each term appearing twice ($A_{\nu i} = A_{i\nu}$). Since matrix A is symmetric the conditions of minimum of capital dissipation with respect to resource's estimates yields the conditions 69), (70). Thus, *in an open microeconomic system that consists of subsystems in internal equilibrium with flows that depend linearly on* *the difference of resources' estimates the resources are distributed in such a way that capital dissipation attains minimum with respect to free variables.* This statement is an analogy of the extremal principle of Prigogine for economic systems.

3.2. Limiting possibilities of economic intermediary

Assume that the system (Figure 5) includes an intermediary that can buy resource from one economic agent and resell it to another. This allows it to extract capital.

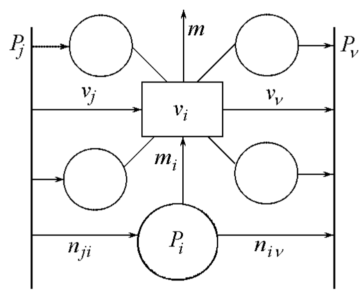

Figure 5. An open system that includes intermediary, markets and passive subsystem.
Figure 0.

Intermediary offers the price v_i during an exchange with the i-th subsystem. The flow of resource here is $m_i(p_i, v_i)$. In a stationary state the problem of extraction of maximal profit takes the following form

$$m = -\sum_{i=1}^{k} m_i(p_i, v_i)v_i \to \max_{v,p} \tag{72}$$

subject to constraints

$$\sum_{i=1}^{k} m_i(p_i, v_i) = 0, \tag{73}$$

$$\sum_{j=1}^{k} n_{ji}(p_j, p_i) = m_i(p_i, v_i), \qquad i = r+1, \ldots, k. \tag{74}$$

Minus in (72) is the result of the assumption that the flow of resource directed from economic agent to the intermediary is positive. This flow is accompanied by reduction of capital. The condition (73) is intermediary's resources' balance and the conditions (74) are resource balances for each of the $k - r$ economic agents.

The Lagrange function of the problem (72), (74) is :

$$L = \sum_{i=1}^{k} \left[m_i(p_i, v_i)(\Lambda - v_i + \lambda_i) - \lambda_i \sum_{j=1}^{k} n_{ji}(p_j, p_i) \right]. \qquad (75)$$

Here $\lambda_i = 0$ for $i \leq r$.

The conditions of optimality have the form

$$\frac{\partial L}{\partial v_i} = 0 \Rightarrow \frac{\partial m_i}{\partial v_i}(\Lambda - v_i - \lambda_i) = m_i(p_i, v_i), \quad i = 1, \dots, k, \qquad (76)$$

$$\frac{\partial L}{\partial p_i} = 0 \Rightarrow \frac{\partial m_i}{\partial p_i}(\Lambda - v_i - \lambda_i) = \lambda_i \sum_{j=1}^{m} \frac{\partial n_{ji}}{\partial p_i}, \quad i = r+1, \dots, k. \qquad (77)$$

The conditions (73), (74), (76), (77) determine $2(k-)$ unknowns p_i and λ_i, and the values of Λ and k optimal prices v_i.

In particular, if $n_{ji} = a_{ji}(p_i - p_j)$, $m_i = a_i(v_i - p_i)$ then these conditions can be rewritten as follows

$$\sum_{i=1}^{m} a_i(v_i - p_i) = 0, \qquad (78)$$

$$\sum_{j=1}^{k} a_{ji}(p_i - p_j) = a_i(v_i - p_i), \quad i = r+1, \dots, k, \qquad (79)$$

$$2v_i = \lambda_i + \Lambda + p_i, \quad i = 1, \dots, k, \qquad (80)$$

$$-a_i(\Lambda - v_i + \lambda_i) = \lambda_i \sum_{j=1}^{k} a_{ji}, \quad i = r+2, \dots, k. \qquad (81)$$

The problem of maximal rate of extraction of capital is a direct analog of the problem of maximal power of heat engine in an open thermodynamic system.

4. Conclusion

Economic systems differ from thermodynamic systems in many respects including voluntary, discretional nature of exchange, production in addition to exchange, competition in various forms, etc. However, thermodynamic and economic systems are both macro-systems which explain many analogies between them, including analogy of irreversibility of processes in them.

Steady state of open thermodynamic and microeconomic systems that include internally equilibrium subsystems with fixed intensive variables (reservoirs) and subsystems whose intensive variables are free (are determined by the exchange flows) were considered. It was shown that for linear dependence of flows on driving forces it corresponds to the conditional minimum of the entropy production (in thermodynamics) or to the conditional minimum of capital dissipation (in microeconomics). The conditions that determine the limiting possibilities of an active subsystem (transformer in thermodynamics and intermediary in microeconomics) were obtained.

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