Journal of Mathematical Sciences, Vol. 195, No. 3, December, 2013

MATHEMATICAL MODELS AND CONTROL FOR SYSTEMS WITH SEGREGATION

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Abstract. We consider mathematical models and optimal control problems for systems that consist of a large number of uncontrolled aggregates and an environment interacting with them. The evolution of each aggregate is described by an ordinary differential equation with a random parameter; the evolution of the environment is defined by the averaged interaction with aggregates and the values of control parameters. We obtain optimality conditions and develop an approach to the calculation of the distribution densities of the residence time of aggregates in the system.

CONTENTS

1. Introduction

In the simplest model of a thermodynamic system, the model of an ideal gas, the components of the system—molecules of the gas—are assumed to be elastic balls in a vacuum that interact with each other. Systems whose elements do not interact with each other directly but influence each other only through the medium (environment) in which the evolution of these elements (aggregates) develops are called systems with full segregation. For brevity, below they are called segregated systems.

Processes of growth and dissolution of crystals, biosynthesis, drying, granulation, fish farming, and etc. are similar to these systems (see [6]).

Segregated model are adequate for systems of social and economic nature, in which the set of elementary economic agents (in economics, they are called households) form a common legal, regulatory, and pricing environment. The state of the environment depends on the interaction with aggregates.

The mathematical features of models of segregated systems are the presence of the averaging on the right-hand side of the differential equations describing the evolution of the state of the environment, and the fact that the control parameters can be applied only to the environment and they change conditions that are common to all aggregates. As for all macrosystems, each separate aggregate in a segregated system cannot be controlled, and there is no way to measure its condition.

The evolution of the state of an aggregate depends on its interaction with the environment and the parameter characterizing the individuality of the aggregate. This parameter is random, and its probability distribution density is known. Typically, a random parameter of an aggregate is a vector,

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Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 82, Nonlinear Control and Singularities, 2012.

one of whose components is the initial value of the state aggregate and the other is its residence time in the system.

2. Systems Isolated with Respect to Aggregates

We consider systems that do not interchange their aggregates with the environment. In processes in such systems, states of the environment and aggregates change in the interval $[0, T]$.

2.1. The model of the system. We denote by x and y the state vectors for the aggregate and the environment, respectively. The evolution of the state of the aggregate is governed by the kinetic equation in which the initial state of each aggregate is a random variable with the probability density $P(\gamma)$:

$$\dot{x}(t,\gamma) = f(x(t,\gamma), y(t)), \quad x(0) = \gamma, \tag{1}$$

where t is the residence time of the aggregate in the system. Therefore, the state $x(t,\gamma)$ is random for any instant t.

The state of the environment at any instant is governed by the equation

$$\dot{y} = \overline{\varphi(y, x(t,\gamma))}^{\gamma} + g(y,t) = \int \varphi(y, x(t,\gamma))P(\gamma)d\gamma + g(y,t), \quad y(0) = y_0. \tag{2}$$

If the system is controlled by factors $u(t)$, then they are contained only on the right-hand sides of Eqs. (2). These equations become

$$\dot{y} = \overline{\varphi(y,u,x)}^{\gamma} + g(y,u,t), \quad y(0) = y_0, \quad u \in V_u, \tag{3}$$

where the set V_u of admissible controls is defined by the restrictions imposed on them at any instant on the control interval $\Delta = [0, T]$. The first term on the right-hand side of (3) characterizes the interaction with aggregates and the second term describes the external influence on the state of the environment; the integral is taken over the domain of the probability density $P(\gamma)$.

In the sequel, we assume that $x(t) \in C^1(\Delta, \mathbb{R} * n)$, $y(t) \in C^1(\Delta, \mathbb{R}^m)$, $u(t) \in C(\Delta, \mathbb{R}^r)$, V_u is a compact set, the function $f : \mathbb{R}^n \to \mathbb{R} * n$ is continuously differentiable with respect to x, and the function $\varphi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable with respect to x y.

2.2. Control problems and conditions of optimality. We write the optimality criterion in the form

$$y_0(T) = \int_0^T [\overline{\varphi_0(y,u,x)}^{\gamma} + g_0(y,u,t)]dt \to \max, \quad y_0(0) = 0. \tag{4}$$

A wide class of optimality criteria can be reduced to this form by the introduction of the appropriate variables.

Necessary conditions of optimality of problem (4), (3), (1) have the following form.

Theorem 1. *Let $(x^*(t,\gamma), y^*(t), u^*(t))$ be an optimal solution. Then there exist multipliers λ_0, $\xi(t) \in C^1(\delta, \mathbb{R}^m)$, and $\psi(\gamma,t) \in C^1(\delta, \mathbb{R}^n)$, which do not vanish simultaneously* (*the multiplier λ_0 can be equal to* 0 *or* 1), *such that the Lagrange function*

$$R = \int \Big\{ \Big[\lambda_0\varphi_0\big(y,u,x(t,\gamma)\big) + \xi\varphi\big(y,u,x(t,\gamma)\big)\Big] + \psi(\gamma,t)f\big(x(t,\gamma),y\big) + \dot{\psi}(\gamma,t)x(t,\gamma)\Big\}P(\gamma)d\gamma + \lambda_0 g_0(y,u,t) + \xi g(y,u,t) + \dot{\xi}y \tag{5}$$

satisfies the following conditions:

(a) *stationarity with respect to $x(t,\gamma)$ for all γ and with respect to $y(t)$*;

(b) *optimality with respect to* $u(t)$.

Formally,

$$\frac{\partial R}{\partial x(t,\gamma)} = 0 \quad \forall\gamma, \quad \frac{\partial R}{\partial y} = 0, \tag{6}$$

$$u^*(t) = \arg\max_{u \in V_u} R(u, y^*(t), x^*(t,\gamma)). \tag{7}$$

Introduce the notation

$$H = \overline{\varphi_0(y,u,x) + \xi\varphi(y,u,x)}^{\gamma} + g_0(y,u,t) + \xi g(y,u,t). \tag{8}$$

Taking into account expression (5) for the nondegenerate solution ($\lambda_0 = 1$), we rewrite the conditions (6) and (7) in the form

$$u^*(t) = \arg\max_{u \in V_u} H(y,u,x), \tag{9}$$

$$\dot{\xi} = -\frac{\partial}{\partial y}\left[H(y,u,x(t,\gamma)) + \overline{\psi(t,\gamma)f(x(t,\gamma),y)}^{\gamma}\right], \tag{10}$$

$$\dot{\psi}(\gamma,t) = -\frac{\partial[\varphi_0(y,u,x) + \xi\varphi(y,u,x) + f(x(t,\gamma),y)]}{\partial x(t,\gamma)}, \tag{11}$$

$$\xi(\tau) = \psi(\gamma,\tau) = 0 \quad \forall\gamma. \tag{12}$$

Here

$$\overline{\psi, f(x(t,\gamma),y)}^{\gamma} = \int \psi(\gamma,t) f(x(t,\gamma),y) P(\gamma) d\gamma. \tag{13}$$

The proof of Theorem 1 is in the Appendix.

3. Systems Open with Respect to Aggregates. The Stationary Regime

Open system interchange with the environment not only by flows affecting the state of the environment, but also by flows of aggregates. We consider only stationary regimes of such systems, in which the state of the environment and the distribution of random variables that affect the status of aggregates is independent of calendar time.

3.1. Mathematical model. In the static regime, the state of the environment y is constant and is equal to the output state since the environment if homogeneous. The state of an aggregate varies depending on its age τ (i.e., time interval from the getting into the system to the present instant) as follows:

$$\frac{dx}{d\tau} = f(x(\gamma,\tau),y), \quad x(0) = \gamma. \tag{14}$$

In some cases, we can obtain a solution of Eq. (14) in the form

$$x = x(\gamma,\tau,y), \tag{15}$$

which allows one to simplify the optimization problem. Solutions of (15) are called kinetic curves.

The age of an aggregate is a random variable. We assume that it is independent of the vector γ and denote its probability density by $P_1(\tau)$. Another random parameter of an aggregate is the residence time τ_f, which is also called the lifetime of the aggregate. The probability density of the residence time is denoted by $P_2(\tau_f)$. We show that the probability densities of the age and the residence time are in a one-to-one correspondence.

A state of the environment is defined by averaged conditions of the form

$$\overline{\varphi(y,u,x(\gamma,\tau))}^{\gamma,\tau} = g(y,u), \tag{16}$$

where u is the control vector and the overline denotes averaging by τ and γ with respect to the probability densities $P_1(\tau)$ and $P_3(\gamma)$. In particular, the right-hand side of Eq. (16) can be equal to $\frac{g}{V}(y - y_0)$, where V is the volume of the system and g is the expenditure rate, which is a control variable.

The dimension of the vector-valued function φ coincides with the dimension of y; the functions f, φ, and the function

$$\overline{\varphi_0(y, u, x(\tau_f))}^{\gamma,\tau_f} \to \max_{u \in V_u} \tag{17}$$

defining the optimality criterion are continuous and continuously differentiable with respect to the totality of their arguments. Some parameters defining the shape of probability distributions of the age and the residence time of aggregates can also be control parameters.

3.2. Optimization of static regimes of systems with segregation. Necessary conditions of optimality of a static regime of a segregated system have the following form.

Theorem 2. *Let $(x^*(\gamma,\tau), y^*, u^*)$ be an optimal solution. Then there exist multipliers λ_0, λ_i, and $\psi(\gamma,\tau) \in C^1$, which do not vanish simultaneously (the multiplier λ_0 can be equal to* 0 *or* 1*), such that the Lagrange functional*

$$S = \lambda_0 \overline{\varphi_0(y, u, x(\gamma, \tau_f))}^{\gamma,\tau_f} + \lambda\Big[\overline{\varphi(y, u, x(\gamma,\tau))}^{\gamma,\tau} - g(y,u)\Big] + \overline{\left[\frac{d\psi(\gamma,\tau)}{d\tau} x(\gamma,\tau) + \psi(\gamma,\tau) f(x(\gamma,\tau), y)\right]}^{\gamma,\tau} \tag{18}$$

satisfies the following conditions:

(a) *the stationarity with respect to $x(\tau,\gamma)$ for all γ and with respect to y;*
(b) *the local unimprovability with respect to $u(t)$.*

Here $P_1(\tau)$ and $P_2(\tau_f)$ are related by Eq. (35) *(see below).*

The proof of Theorem 2 is in the Appendix.

Formally, the necessary conditions of optimality can be written as follows:

$$\frac{\partial R}{\partial x} = 0, \quad \frac{\partial S}{\partial u}\delta u \le 0, \quad \frac{\partial S}{\partial y} = 0, \tag{19}$$

where δu is an admissible variation of controls with account of imposed restrictions $u \in V_u$.

For a functional S of the form (18), the conditions (19) have the form

$$\begin{cases} \dfrac{d\psi}{d\tau} = -\dfrac{\partial}{\partial x}\left[\psi(\gamma,\tau) f(x,y) + \lambda\varphi(y,u,x)\right], \\ \psi(\gamma,\tau_f) = \dfrac{\partial}{\partial x(\gamma,\tau_f)}\varphi_0(y,u,x), \end{cases} \tag{20}$$

$$\frac{\partial}{\partial u}\left[\overline{\varphi_0(y,u,x(\gamma,\tau))}^{\gamma,\tau_f} + \lambda\overline{\varphi(y,u,x(\gamma,\tau))}^{\gamma,\tau}\right]\delta u \le 0, \tag{21}$$

$$\frac{\partial}{\partial y}\left[\overline{\varphi_0(y,u,x(\gamma,\tau_f))}^{\gamma,\tau_f} + \overline{\Big[\lambda\varphi\big(y,u,x(\gamma,\tau)\big) + \psi(\gamma,\tau) f\big(x(\gamma,\tau),y\big)\Big]}^{\gamma,\tau}\right] = 0. \tag{22}$$

These conditions together with Eqs. (14) and the averaged conditions (16) define the vectors u, y, and λ and the functions $x(\gamma,\tau)$ and $\psi(\gamma,\tau)$.

The conditions of optimality can be simplified if we obtain a kinetic curve $x(\gamma,\tau,y)$. In this case, the problem is reduced to the form

$$I = \overline{\varphi_0(x(\gamma,\tau_f,y),u,y)}^{\gamma,\tau_f} \to \max, \tag{23}$$

$$J = \overline{\varphi(x(\gamma,\tau,y),u,y)}^{\gamma,\tau} - g(y,u) = 0. \tag{24}$$

Averaging is performed here with respect to τ; for the function φ_0 we use the distribution P_2 of the residence time of aggregates, and for the function φ we use the distribution P_1 of the age of aggregates.

Problem (23), (24) is a nonlinear-programming problem. A necessary condition of optimality of its solution for functions φ_0 and φ continuously differentiable with respect to y and u is as follows: there exists a nonzero vector $\lambda = (\lambda_0, \lambda_1, \dots)$ for which the Lagrange function $S = \lambda_0 I + \lambda J$ on the optimal solution is stationary with respect to y and is locally unimprovable with respect to $u \in V_u$:

$$\frac{\partial}{\partial u}\left[\overline{\varphi_0(x(\gamma,\tau_f,y),u,y)}^{\gamma,\tau_f} + \lambda(\overline{\varphi(x(\gamma,\tau,y),u,y)}^{\gamma,\tau} - g(y,u))\right]\delta u \le 0, \tag{25}$$

$$\left(\frac{\partial}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial}{\partial y}\right)\left[\overline{\varphi_0\big(x(\gamma,\tau_f,y),u,y\big)}^{\gamma,\tau_f} + \lambda\Big(\overline{\varphi\big(x(\gamma,\tau,y),u,y\big)}^{\gamma,\tau} - g(y,u)\Big)\right] = 0. \tag{26}$$

Example 1 (optimal choice of the average residence time of aggregate in a system)**.** Let a system be homogeneous by environment and by aggregates. Let

$$x(\tau,y) = \tau^2 e^{-y\tau} \tag{27}$$

be the kinetic curve and

$$P_1(\tau,\Theta) = P_2(\tau,\Theta) = \frac{1}{\Theta}e^{-\tau/\Theta} \tag{28}$$

be the distribution densities. The criterion of optimality is

$$I = \frac{1}{\Theta}\int\limits_0^\infty P_2(\tau,\Theta)x(\tau,y)d\tau \to \max, \tag{29}$$

$$J = \int\limits_0^\infty P_1(\tau,\Theta)\big[y - x(\tau,y)\big]d\tau = 0, \quad \Theta \ge 0. \tag{30}$$

The average residence time $\Theta = \frac{V}{g}$ for aggregates in the system must be found.

The conditions of optimality (25) and (26) for a nondegenerate solution take the form

$$\frac{\partial S}{\partial \Theta} = 0 \to \int\limits_0^\infty \tau^2 e^{-\tau}\left(y + \frac{1}{\Theta}\right)\left[\left(\frac{\tau}{\Theta} - 2\right)\frac{1}{\Theta^2} + \lambda(y - \tau^2 e^{-y\tau})\right]d\tau = 0, \tag{31}$$

$$\frac{\partial S}{\partial y} = 0 \to \int\limits_0^\infty e^{-\tau/\Theta}\left[\frac{1}{\Theta}(1 + \tau^3 e^{-y\tau}) + \lambda(1 + \tau^2 e^{-y\tau})\right]d\tau = 0, \tag{32}$$

which, together with the condition (30), define λ, y, and Θ.

4. Distributions Densities for the Residence Time of Aggregates

The residence time of aggregates is one of the most important parameters common to all segregated systems. Consider the method of calculation of the distributions densities of the residence time.

4.1. Relation between the distributions of the residence time and the age of aggregates in the system. Assume that the initial system x_0 is fixed and the residence time of aggregates in the system (age) τ and the residence time τ_f of aggregates at the output of the system (lifetime) are random. These random variables are related and hence their distribution densities $P_1(\tau)$ and $P_2(\tau_f)$ are also related.

It is important to find this relation since the distribution of the residence time $P_2(\tau_f)$ defines the properties of the output flow, while the distribution of age $P_1(\tau)$ defines the kinetics of the interaction of aggregates and the environment inside the system; moreover, in many cases, the distribution $P_2(\tau_f)$

can be found experimentally using tracers [1] when one supplies a portion of aggregates to the input of the system and measures the portion $P_2(\tau_f)$ of aggregates leaving the system. The distribution of the age $P_1(\tau)$ must be calculated if $P_2(\tau_f)$ is known.

Let P_2 be given. The fraction of aggregates that leaves the system up to the instant τ in the stationary case is

$$F(\tau) = \int\limits_0^\tau P_2(\tau_f)d\tau_f.$$

The value of the distribution density is equal to the fraction of aggregates with age τ remaining in the system. We obtain

$$P_1(\tau) = \frac{1 - \int\limits_0^\tau P_2(\tau_f)d\tau_f}{\int\limits_0^\infty \left(1 - \int\limits_0^\tau P_2(\tau_f)d\tau_f\right) d\tau}. \tag{33}$$

We show that the denominator of this relation is equal to the average residence time of the aggregates in the system:

$$\Theta = \int\limits_0^\infty \tau_f P_2(\tau_f)d\tau_f. \tag{34}$$

Indeed, the integral in the denominator of (33) is equal to the limit (as $s \to 0$) of the Laplace image of the integrand

$$\lim_{s\to 0} L\left[1 - \int\limits_0^\tau P_2(\tau_f)d\tau_f\right] = \lim_{s\to 0} \frac{1}{s}\left(1 - P_2(s)\right).$$

By L'Hôpital's rule, we obtain that this limits is equal to the limit as $s \to 0$ of $-\dfrac{dP_2(s)}{ds}$, which, in its turn, is equal to the integral in (34); hence

$$P_1(\tau) = \frac{1}{\Theta}\left(1 - \int\limits_0^\tau P_2(\tau_f)d\tau_f\right). \tag{35}$$

For any segregated system, this relation allows one to find the distribution density of the residence time of aggregates in the system using the distribution density of the residence time of aggregates leaving the system, in the stationary regime.

We rewrite Eq. (35) using the Laplace transform:

$$P_1(s) = \frac{1}{\Theta s}\left(1 - P_2(s)\right). \tag{36}$$

The distributions $P_2(\tau_f)$ and $P_1(\tau)$ are the same in the case where

$$P_2(\tau_f) = \frac{1}{\Theta}e^{-\frac{\tau_f}{\Theta}}.$$

Indeed, in this case

$$P_2(s) = \frac{1}{\Theta s + 1}.$$

By formula (36) we have

$$P_1(s) = \frac{1}{\Theta s}\left(1 - \frac{1}{\Theta s + 1}\right) = \frac{1}{\Theta s + 1}.$$

The Laplace transform allows one to find the distribution densities of the residence time of aggregates in systems with an arbitrary structure.

4.2. Structure analysis of the distributions of the residence time and the age of aggregates. A segregated system can consist of several subsystems that interchange by flows of aggregates. In each subsystem, the state of the environment depends on the control factors and the states of aggregates averaged by their age. The relation between the distribution densities for different structures allows one to use experimental data obtained for any point of the system for the calculation of parameters of other subsystems.

In the sequel, we consider the distribution $P_2(\tau_f)$ of the residence time τ_f. We denote its Laplace image by $P_2(s)$. The distribution of the age of aggregates $P_1(\tau)$ and its Laplace image $P_1(s)$ can be calculated by the formulas (35) and (36) for a simple system; formulas for the complex system will be obtained below.

Simplest models and elementary operations In the case of ordered motion of aggregates from the input to the output of the system (hydrodynamic extrusion regime, queue), the residence times τ_f^0 of all aggregates in the system are the same:

$$P_2(\tau_f) = \delta(\tau_f - \tau_f^0), \quad P_2(s) = e^{-s\tau_f^0}. \tag{37}$$

In the case of uniform distribution of aggregates in the system (hydrodynamics of ideal mixing), we have

$$P_2(\tau_f) = \frac{1}{\Theta} e^{-\tau_f/\Theta}, \quad P_2(s) = \frac{1}{\Theta s + 1}, \tag{38}$$

where Θ is the average residence time of aggregates in the system; it is equal to the ratio of the number of aggregates to its expenditure. Since the fractions of aggregates in the space of the system and in the output flow are the same, we see that Θ is the ratio of the part of the space of the system occupied by aggregates to the expenditure.

In the system, the branching and merging of flows can occur. In this case, the distribution densities of the residence times change.

Merging of flows. Let for any of n flows with expenditures g_i, $i = 1, \dots, n$, the distribution $P_{2i}(\tau_f)$ be known. Introduce the notation

$$\gamma_i = \frac{g_i}{\sum\limits_{j=1}^{n} g_j}, \quad \gamma_i \geq 0, \quad \sum_{i=1}^{n} \gamma_i = 1.$$

The fraction of aggregates in the ith flow that are in the system during the time from τ_f to $\tau_f + d\tau_f$ is equal to $g_i P_{2i}(\tau_f) d\tau_f$. This fraction in the flow after the merging is

$$g P_2(\tau_f) d\tau_f = \sum_{i=1}^{n} g_i P_{2i}(\tau_f) d\tau_f.$$

This implies

$$P_2(\tau_f) = \sum_{i=1}^{n} \gamma_i P_{2i}(\tau_f). \tag{39}$$

When flows branch, we have

$$P_{2i}(\tau_f) = P_2(\tau_f) \quad \forall i. \tag{40}$$

Seires connection of subsystems. Let two subsystems be connected in series. Then for the first subsystem, the distribution of the residence time $P_{21}(\tau_f)$ and the distribution of the age $P_{11}(\tau)$ are related by the expression (35). For the second subsystem, we will distinguish between the age of aggregates in the system τ and their age only in the second subsystem τ_2. Moreover, we will distinguish between the residence time τ_f in the output of the system and the residence time τ_{fi} in each of the subsystems.

Since

$$\tau_f = \tau_{f1} + \tau_{f2}$$

and these two random variables are independent, the distribution density of the sum is equal to the convolution of the distribution densities of the summands:

$$P_2(\tau_f) = P_{21}(\tau_{f1}) * P_{22}(\tau_{f2}).$$

After the Laplace transform, the convolution converts in the product and we have

$$P_2(s) = P_{21}(s)P_{22}(s). \tag{41}$$

The distribution of the age of aggregates for the first subsystem in the domain of the transforms is

$$P_{11}(s) = \frac{1}{\Theta_1 s}(1 - P_{21}(s)). \tag{42}$$

For the second subsystem, the age of aggregates in the system is $\tau = \tau_{f1} + \tau_2$, and its distribution is

$$P_1(\tau) = P_{21}(\tau_{f1}) * P_{12}(\tau_2).$$

In the domain of the transforms

$$P_1(s) = P_{21}(s) * P_{12}(s). \tag{43}$$

This distribution is related to the distribution of the residence time in the system by the relation (36):

$$P_1(s) = \frac{1}{(\Theta_1 + \Theta_2)s}(1 - P_2(s)). \tag{44}$$

Therefore, for the distribution of the age of aggregates in the second subsystem we have

$$P_{12}(s) = \frac{1}{(\Theta_1 + \Theta_2)s}\left(\frac{1}{P_{21}(s)} - P_{22}(s)\right). \tag{45}$$

Calculation of moments of distribution densities. Many characteristics of the distribution densities of the residence time and the age (mean values, variance, etc.) can be calculated by using their Laplace images without transition to originals. For example, the mean value of the age τ and its variance are as follows:

$$\Theta_i = \lim_{s\to 0}\left[-\frac{d}{ds}P_i(s)\right], \tag{46}$$

$$D_i = \lim_{s\to 0}\left[\frac{d^2 P_i(s)}{ds^2} - \left(\frac{dP_i(s)}{ds}\right)^2\right]. \tag{47}$$

These expressions allow one, by using the Laplace transform, to find the distribution densities of the residence time and the age of aggregates in each of subsystems for a system of arbitrary structure, including subsystems connected in parallel, subsystems with recycling of aggregates, etc.

Example 2. Find the distribution density of the residence time in a system with recycling consisting of two subsystems. The flow of aggregates passes through the system whose distribution density of the residence time $P_{a2}(\tau_f)$ is known:

$$P_{ar}(\tau_f) = \frac{1}{\Theta}e^{-\frac{\tau_f}{\Theta}}, \quad P_{a2}(s) = \frac{1}{\Theta s + 1}. \tag{48}$$

After this, the fraction r of the output flow of aggregates returns to the input of the system through the subsystem , for which

$$P_{b2}(\tau_f) = \delta(\tau_f - \tau_0), \quad P_{b2}(s) = e^{-\tau_0 s}. \tag{49}$$

Here Θ and τ_0 depend on the expenditure and correspond to the expenditure g at the input of the system. Since the expenditure through the recycle is equal to rg and the expenditure through the subsystem is equal to $g(1+r)$, we have

$$P^r_{b2}(s) = e^{-\frac{\tau_0 s}{r}}, \quad P^r_{a2}(s) = \frac{r+1}{\Theta s + r + 1}. \tag{50}$$

We find $P_2(\tau_f)$ for the whole system.

We use the Laplace transform. In the domain of the transform, after merging of flows at the input of the subsystem A, taking into account (50), we have

$$P_{abx}(s) = \frac{1 + rP_2(s) \cdot e^{-\frac{s\tau_0}{r}}}{1+r}. \tag{51}$$

On the other hand,

$$P_2(s) = P_{abx}(s) \cdot \frac{r+1}{\Theta s + r + 1}. \tag{52}$$

Substituting this in (51), we obtain

$$P_2(s)(\Theta s + 1 + r) = 1 + rP_2(s)e^{-\frac{\tau_0 s}{r}},$$

and hence the Laplace image of the distribution density of the residence time τ_f of aggregates in the system is

$$P_2(s) = \frac{1}{\Theta s + r\left(1 - e^{-\frac{\tau_0 s}{r}}\right) + 1}. \tag{53}$$

5. Appendix

Proof of Theorem 1. Necessary conditions of optimality of segregated control systems that are characterized by the model (1), (3), (4) are consequences of the maximum principle for the optimal control problem with scalar argument in the canonical form. This problem has the form

$$I = \int_0^T \left[f_{01}(t, x(t), u(t)) + \sum_l f_{02}(t, x(t))\delta(t - t_l) \right] dt \to \max \tag{54}$$

under the conditions

$$\begin{gathered} J_j(\tau) = \int_0^T \Big[f_{j1}(t, x(t), u(t), \tau) + f_{j2}(t, x(t), \tau)\delta(t - \tau) \Big] dt = 0, \\ \forall \tau \in [0, T], \quad j = \overline{1, m}, \quad u \in V_u, \end{gathered} \tag{55}$$

where $u(t)$ and $x(t)$ are piecewise continuous and piecewise linear vector-valued functions, respectively, the values of $u(t)$ belong to a closed bounded domain V of the space $\mathbb{R}^n$, the functions f_{j1} and f_{j2}, $j = \overline{0, m}$, are defined on the direct product of the sets of admissible values of their arguments and are continuously differentiable with respect to x and t, and the functions f_{j1} are continuous in u. The functional I is bounded on the set of admissible solutions.

Note that by $u(t)$ we denote only the variables that are included in the functions f_{j1} for $j = 0, 1, \ldots, m$. For brevity, we call them the variables of the first group.

An optimal solution of this problem (if it exists) is described by the following theorem.

Theorem 3 (maximum principle for problems in the canonical form, [3]). *Let $u^*(t), x^*(t)$ be a desired solution. Then the following conditions hold*:

(1) *There exists a scalar $\lambda_0 \geq 0$ and a vector-valued function $\lambda(\tau) = (\lambda_1(\tau), \ldots, \lambda_m(\tau))$, which is continuously differentiable for almost all t, does not vanish simultaneously with λ_0 on the segment $[0, T]$, and vanish outside this segment, such that for the functional*

$$S = \lambda_0 I + \sum_{j=1}^{m} \int_0^T \lambda_j(\tau) J_j(\tau) d\tau = \int_0^T R dt \tag{56}$$

and its integrand

$$R = \lambda_0 R_0 + \sum_{j=1}^{m} R_j^{CB}, \tag{57}$$

where

$$R_0 = f_{01}(t, x(t), u(t)) + \sum_l f_{02}(t, x(t)) \delta(t - t_l)$$

and

$$R_j^{CB} = \int_0^T \lambda_j(\tau) \Big[f_{j1}(t, x(t), u, \tau) + f_{j2}(t, x(t), \tau) \delta(\tau - t) \Big] d\tau, \tag{58}$$

the following relations hold:

$$\frac{\delta R}{\delta x} = 0, \tag{59}$$

$$u^*(t) = \arg\max_{u \in V_u} R(x, \lambda, u). \tag{60}$$

(2) *If the desired solution contains a vector of parameters $a \in V_a$ that are constant on the interval $[0, T]$, then the optimality conditions* (59) *and* (60) *must be complemented by the conditions of the local unimprovability* (*with respect to a*) *of the functional S*:

$$\frac{\partial S}{\partial a} \partial a \leq 0, \tag{61}$$

where ∂a are variations of the parameters a admissible with respect to the inclusion $a \in V_a$.

Thus, for the problem in the canonical form (54), (55), the conditions of the maximum principle (59) and (60)) are valid, in which the term R_0 in the function R corresponds to the optimality criterion, and the terms R_j^{CB}, $j = 1, 2, \ldots, m$, correspond to the summands.

To make these conditions for segregated systems more specific, we must, for each relation (1), (3), (4), after the reduction to the canonical form, write the terms R_0 and R_j^{CB} and substitute them in (59) and (60)). In [3], these terms are obtained for conditions in the form of differential equations and integral criteria. In the notation used for segregated systems, the terms of the function R take the form

$$R_0 = \overline{\varphi_0(y, u, x)}^{\gamma} + g_0(y, u, t), \tag{62}$$

$$R_y^{CB} = \xi \overline{\varphi(y, u, x(t, \gamma))}^{\gamma} + \xi g(y, u, t) + \dot{\xi} y, \tag{63}$$

$$R_x^{CB} = \overline{\psi(\gamma, t) f(x(t, \gamma), y) + \dot{\psi}(\gamma, t) x(t, \gamma)}^{\gamma}. \tag{64}$$

The control parameters are variables of the first group.

Therefore, the function R becomes

$$R = \overline{[\lambda_0\varphi_0(y,u,x(t,\gamma)) + \xi\varphi(y,u,x(t,\gamma)) + \psi(\gamma,t)f(x(t,\gamma),y) + \dot{\psi}(\gamma,t)x(t,\gamma)]}^{\gamma} + \lambda_0 g_0(y,u,t) + \xi g(y,u,t) + \dot{\xi}y. \quad (65)$$

For this function, the conditions of stationarity with respect to x and y and the optimality with respect to u coincide with the conditions of Theorem 1.

Proof of Theorem 2. The terms of the Lagrange functional S for the conditions (14), (16), and (17)) have the form (see [3])

$$\begin{aligned}
S_1^{CB} &= \overline{\left[\frac{d\psi(\gamma,\tau)}{d\tau}x(\gamma,\tau) + \psi(\gamma,\tau)f\big(x(\gamma,\tau),y\big)\right]}^{\gamma,\tau}, \\
R_1^{CB} &= \left[\frac{d\psi(\gamma,\tau)}{d\tau}x(\gamma,\tau) + \psi(\gamma,\tau)f\big(x(\gamma,\tau),y\big)\right], \\
S_2^{CB} &= \lambda\Big[\overline{\varphi\big(y,u,x(\gamma,\tau)\big)}^{\gamma,\tau} - g(y,u)\Big], \\
S_0 &= \overline{\varphi_0\big(y,u,x(\gamma,\tau_f)\big)}^{\gamma,\tau_f}.
\end{aligned} \quad (66)$$

The conditions of the stationarity of R with respect to x, the stationarity of S with respect to y, and the local unimprovability of S with respect to u, by Theorem 3, lead to the relations (20), (21), and (22).

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